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HYDRODYNAMICS WITH QUADRATIC PRESSURE.
2. EXAMPLES

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Exact solutions of Euler equations that describe the motion of an ideal incompressible fluid with quadratic pressure are studied. The solutions are described by explicit formulas and can be physically interpreted. The dynamics of a spherical fluid volume is studied for specified initial velocity fields. It is shown that under certain initial conditions, the spherical volume can evolve into a torus-shaped body, thereby changing the connectivity of the region occupied by the fluid.

Introduction. The present paper is a continuation of [1], in which we give an algorithm for integrating and the general properties of solutions of the Euler equations

$$D\boldsymbol{u} + \nabla p = 0, \qquad \operatorname{div} \boldsymbol{u} = 0 \tag{1}$$

with quadratic pressure

$$p = k(t)(x^2 + y^2 + z^2)/2. \tag{2}$$

In this paper, we use the notation of [1].

If the Jacobian matrix $J = \partial \boldsymbol{u}/\partial \boldsymbol{x}$ has an eigenvalue with a multiplicity of 2, the elliptic functions of time defining the flow dynamics reduce to rational functions. The Lamé equation describing fluid particle trajectories becomes an equation with a rational potential. The solutions are written in elementary functions but describe nontrivial motion of the fluid. If we specify the initial velocity fields $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x}_0)$ such that the matrix $J_0 = \partial \boldsymbol{u}_0/\partial \boldsymbol{x}_0$ has constant algebraic invariants and $\operatorname{tr} J_0 = 0$, we obtain examples of exact solutions of the Euler equations with pressure in the form (2). A common feature of these solutions is the presence of a singularity in motion — degeneration of the dimension of the region occupied by the fluid.

Integration of the Equations of Trajectories. Let the Jacobian matrix J have an eigenvalue $\lambda = \lambda(t)$ with a multiplicity of 2 and a basis of its eigenvectors exist in $\mathbb{R}^3(\boldsymbol{x})$. Then, as is shown in [1], we have

$$\lambda_1 = \lambda_2 = \lambda = \lambda_0(1 + \lambda_0(t_0 - t))^{-1}, \qquad \lambda_3 = -2\lambda, \tag{3}$$

where $\lambda_0 = \lambda(t_0)$ is an arbitrary real number. In this case,

$$k = 2k_2/3 = -(1/3)\operatorname{tr} J^2 = -2\lambda^2.$$

The equations of trajectories become

$$\frac{d^2\boldsymbol{x}}{dt^2} - \frac{2\lambda_0}{(1 + \lambda_0(t_0 - t))^2}\,\boldsymbol{x} = 0. \tag{4}$$

The fundamental system of solutions of each equation in system (4) consists of the functions

$$q_1 = \tau^{-1}, \qquad q_2 = \tau^2, \tag{5}$$

where $\tau = 1 + \lambda_0(t_0 - t)$. The Wronskian of the system of solutions (5) is a constant $W(t) = W_0 = -3$. The general solution of Eqs. (4) subject to the initial data

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$$\boldsymbol{x}|_{t=0} = \boldsymbol{x}_0, \qquad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0(\boldsymbol{x}_0)$$

is written as

$$\boldsymbol{x} = [\lambda_0(2q_1 + q_2)\boldsymbol{x}_0 + (q_1 - q_2)\boldsymbol{u}_0(\boldsymbol{x}_0)]/(3\lambda_0). \tag{6}$$

For all solutions, the pressure has the form

$$p = p_0(t) - \lambda_0^2\tau^{-2}r^2,$$

where $p_0(t)$ is an arbitrary function such that $p \geqslant 0$ for all times. Obviously, $p \to \infty$ as $\tau \to 0$.

The singular points of the solution of the linear system (4) are the singular point of the coefficients [singularity of the eigennumber of (3)] and an infinitely distant point: $\tau = 0$, $t = t_0 + \lambda_0^{-1}$ and $\tau = \infty$, $t = \infty$. Then, solution (6) can be considered in the half-intervals

$$T_1\colon \quad -\infty < t < t_0 + \lambda_0^{-1}, \qquad T_2\colon \quad t_0 + \lambda_0^{-1} < t < +\infty. \tag{7}$$

We consider several examples of solutions with different initial velocity fields $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x}_0)$.

Example 1. The initial velocity field is defined by the formulas

$$u_0 = -2x_0 + 3U(y_0, z_0), \qquad v_0 = y_0, \qquad w_0 = z_0 \tag{8}$$

with an arbitrary smooth function U of their arguments ($\lambda_0 = 1$). The equations of trajectories (6) for the initial data (8) take the form

$$x = U\tau^{-1} + (x_0 - U)\tau^2, \qquad y = \tau^{-1}y_0, \qquad z = \tau^{-1}z_0, \tag{9}$$

where $\tau = 1 + t_0 - t$.

The motion of each fluid particle occurs in the plane

$$\Pi\colon\ z_0y - y_0z = 0 \tag{10}$$

and is defined by the cubic curve

$$y^2x = y_0^{-1}Uy^3 + (x_0 - U)y_0^2. \tag{11}$$

According to Newton's classification of cubic curves [2], curve (11) belongs to the fourth group (hyperbolic conic sections). A special feature of curves in the form of (11) is that in plane (10) there are two asymptotes: $y = 0$ (multiple of 2) and $y = y_0U^{-1}x$. Definition of the function $U = U(y_0, z_0)$ determines the position of the curve on the plane and the slope of the asymptote. Variation of the function U does not change the type of the curve.

The fluid motion described in Lagrangian coordinates by formulas (9) is written in Euler coordinates as

$$u = 3\tau^{-2}U - 2\tau^{-1}x, \qquad v = \tau^{-1}y, \qquad w = \tau^{-1}z, \tag{12}$$

where $U = U(y\tau, z\tau)$.

The Jacobian matrix of the velocity field (12) has the form

$$J = \begin{bmatrix} -2\tau^{-1} & -3\tau^{-1}U_1 & -3\tau^{-1}U_2 \\ 0 & \tau^{-1} & 0 \\ 0 & 0 & \tau^{-1} \end{bmatrix}, \tag{13}$$

where U_i $(i = 1, 2)$ are partial derivatives of the function U with respect to its arguments. The matrix J (13) has eigenvalues $\lambda_{1,2} = \tau^{-1}$ and $\lambda_3 = -2\tau^{-1}$; the logarithmic potentials of the eigenvalues are $q_{1,2} = \tau^{-1}$ and $q_3 = \tau^2$. We can choose the following right eigenvectors:

$$\boldsymbol{r}_1 = 3(-U_1U_2, U_2, 0)^{\mathrm{T}}, \qquad \boldsymbol{r}_2 = 3(U_1U_2, 0, -U_1)^{\mathrm{T}}, \qquad \boldsymbol{r}_3 = (1, 0, 0)^{\mathrm{T}},$$

where the superscript "T" denotes transposition of the vector ($\boldsymbol{r}_i$ are column vectors). In this case, the representation for the vortex $\boldsymbol{\omega}$ [1] has the form

$$\boldsymbol{\omega} = \tau^{-1}(\boldsymbol{r}_1 + \boldsymbol{r}_2) + 0 \cdot \tau^2\boldsymbol{r}_3 = 3(0, \tau^{-1}U_2, -\tau^{-1}U_1)^{\mathrm{T}}.$$

A fluid volume that is initially bounded by the sphere $x_0^2 + y_0^2 + z_0^2 = 1$ evolves with time into a volume bounded by the surface

$$\tau^2(y^2 + z^2) + (x - U\tau^{-1})^2\tau^{-4} + 2U(x - U\tau^{-1})\tau^{-2} + U^2 = 1,$$

where $U = U(y\tau, z\tau) = U(y_0, z_0)$.

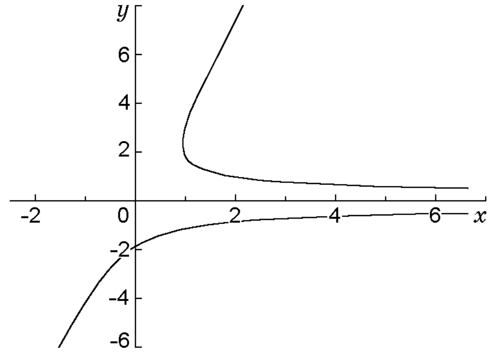

Fig. 1

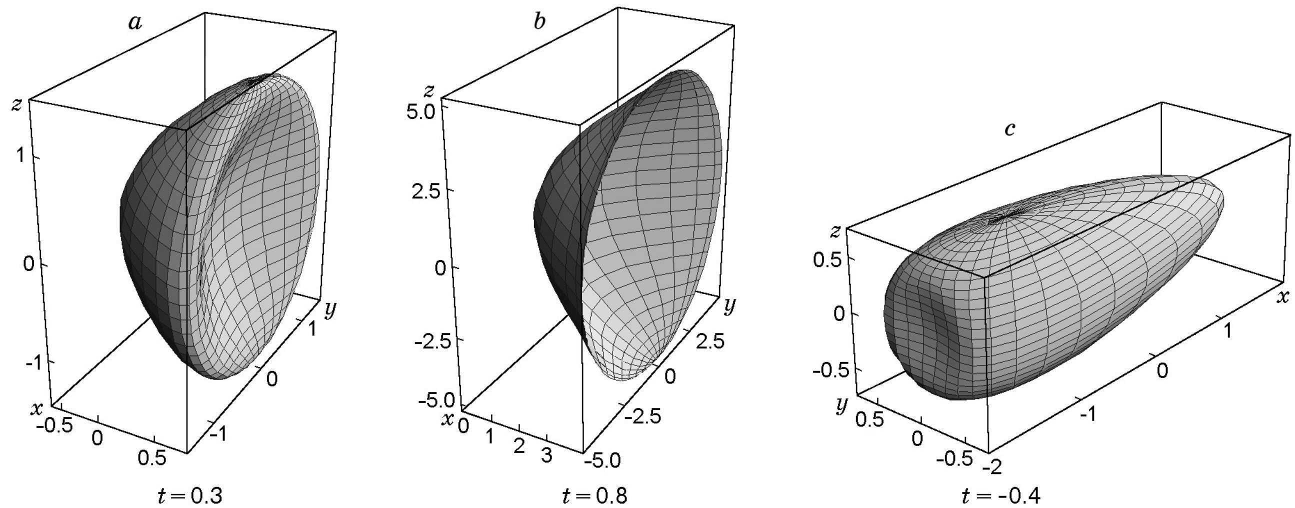

Fig. 2

Figures 1 and 2 show the trajectories (11) and dynamics of the fluid sphere for $U = \sin(y_0^2 + z_0^2)$ and $t_0 = 0$. Figure 1 shows a trajectory of a fluid particle in the plane $z_0 = 10$ for $(x_0, y_0) = (1, 3)$. Various branches of curve (11) correspond to the motion in the half-intervals T_1 and T_2 of type (7). As $y_0 > 0$, the upper branch of curve (11), on which $y > 0$, corresponds to values $t < 1$, and the lower branch, on which $y < 0$, has values $t > 1$. These branches correspond to motions of different asymptotic forms: on the upper branch, $y \to +0$ as $t \to -\infty$ and on the lower branch, $y \to -0$ as $t \to +\infty$.

Figure 2 shows the evolution of the spherical volume with approach to different singular points: $t \to 1$ (Fig. 2a and b) and $t \to -\infty$ (Fig. 2c). Special features of the motion are the loss of dimensionality and the degeneration of the region. The degeneration patterns are different. Thus, as $t \to 1$, the sphere flattens out to an almost flat manifold, and as $t \to \infty$, the sphere stretches to a needle. Similar special features were found and studied by Ovsyannikov [3] for motion with a linear velocity field.

Example 2. In Cartesian coordinates, the initial velocity field has the form

$$u_0 = x_0 + q_0 \cos(bz_0), \qquad v_0 = y_0 - q_0 \sin(bz_0), \qquad w_0 = -2z_0, \tag{14}$$

where q_0 and b are arbitrary real parameters. In the cylindrical coordinates $x_0 = r\cos\psi$ and $y_0 = r\sin\psi$, the vector field (14) has the form

$$V_{0c} = r + q_0 \cos(\psi + bz_0), \qquad W_{0c} = -q_0 \sin(\psi + bz_0), \qquad w_0 = -2z_0,$$

where V_{0c} and W_{0c} are the radial and circular velocity components. For the vector field (14), $\lambda_0 = 1$ such that $\tau = 1 + t_0 - t$. The equations of trajectories (6) are represented by the equations

$$x = \tau^{-1}x_0 + (q_0/3)(\tau^{-1} - \tau^2)\cos(bz_0), \quad y = \tau^{-1}y_0 + (q_0/3)(\tau^{-1} - \tau^2)\sin(bz_0), \quad z = \tau^2 z_0.$$

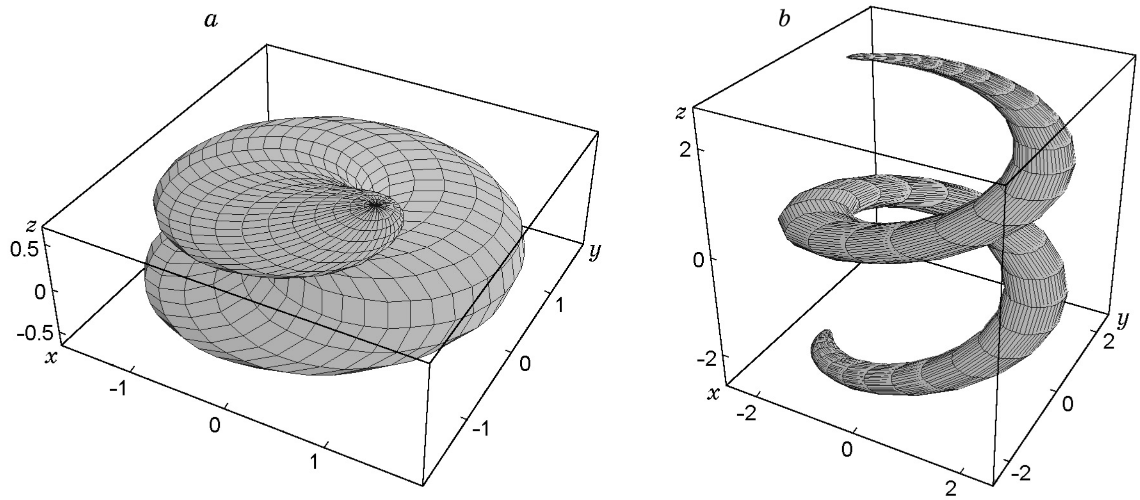

Fig. 3

A plane Π in which a fluid particle with initial data $[\boldsymbol{x}_0, \boldsymbol{u}_0(\boldsymbol{x}_0)]$ in the form of (14) moves is defined by the equation

$$(x - \tau^{-1}x_0)\sin(bz_0) - (y - \tau^{-1}y_0)\cos(bz_0) = 0$$

into which we substitute $\tau = |z/z_0|^{1/2}$ for $z_0 \neq 0$. If $z_0 = 0$, the motion occurs in the plane $z = 0$. The dynamics of fluid particles that, at $t_0 = 0$, occupied a volume bounded by a sphere $x_0^2 + y_0^2 + z_0^2 = a^2$ ($a = \text{const}$) is described by the equations

$$x = a\tau^{-1}\sin\theta\cos\varphi + (q_0/3)(\tau^{-1} - \tau^2)\cos(ab\cos\theta),$$
$$y = a\tau^{-1}\sin\theta\sin\varphi + (q_0/3)(\tau^{-1} - \tau^2)\sin(ab\cos\theta), \qquad z = a\tau^2\cos\theta, \tag{15}$$

where the standard spherical coordinates a, θ, and φ ($0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \varphi \leqslant 2\pi$) define the boundary of the region at any time in the half-intervals (7).

Figure 3 show the evolution of surface (15) at various times with approach to one of the singular points: $t \to 1$ (a) and $t \to -\infty$ (b). In this case, $a = 1$, $b = 6$, and $q_0 = 3$. In Fig. 3a, $t = 0.2$, and as $t \to 1$, the height of the spiral turn tends to zero. In Fig. 3b, $t = -0.6$, and as $t \to -\infty$, the sphere evolves into a spiral arc, which becomes thin and develops in space with time. The volume of the indicated regions is preserved by virtue of incompressibility of the fluid and it is equal to the volume of the initial sphere. In this example, the singularity of the motion is also manifested in degeneration of the dimensionality of the region.

Example 3. In Cartesian coordinates, the initial velocity field has the form

$$u_0 = x_0 - \sigma r^{-1}y_0z_0, \qquad v_0 = y_0 + \sigma r^{-1}x_0z_0, \qquad w_0 = -2z_0, \tag{16}$$

where $r = \sqrt{x_0^2 + y_0^2}$ and σ is a real parameter. In the cylindrical coordinates $x_0 = r\cos\psi$ and $y_0 = r\sin\psi$, the vector field (16) takes the form

$$V_{0c} = r, \qquad W_{0c} = \sigma z_0, \qquad w_0 = -2z_0, \tag{17}$$

where V_{0c} and W_{0c} are the radial and circumferential velocity components in the plane Ox_0y_0, and w_0 is the velocity component along the axis Oz_0.

The equations of trajectories (6) take the form

$$x = \tau^{-1}x_0 - \sigma(\tau^{-1} - \tau^2)z_0\sin\psi, \quad y = \tau^{-1}y_0 + \sigma(\tau^{-1} - \tau^2)z_0\cos\psi, \quad z = \tau^2 z_0. \tag{18}$$

The trajectory of a fluid particle that starts at $t_0 = 0$ from the point x_0 with a velocity $\boldsymbol{u}_0(\boldsymbol{x}_0)$ (16) lies in the plane

$$(x - \tau^{-1}x_0)\cos\psi + (y - \tau^{-1}y_0)\sin\psi = 0. \tag{19}$$

We should substitute $\tau = |z/z_0|^{1/2}$ into Eq. (19). A surface consisting of trajectories of the fluid particles can be constructed in the following manner: Excluding the quantity τ from Eqs. (18), we obtain the relation

$$(x - |z_0/z|^{1/2}x_0)^2 + (y - |z_0/z|^{1/2}y_0)^2 = \sigma^2 z_0^2(|z_0/z|^{1/2} - z/z_0)^2. \tag{20}$$

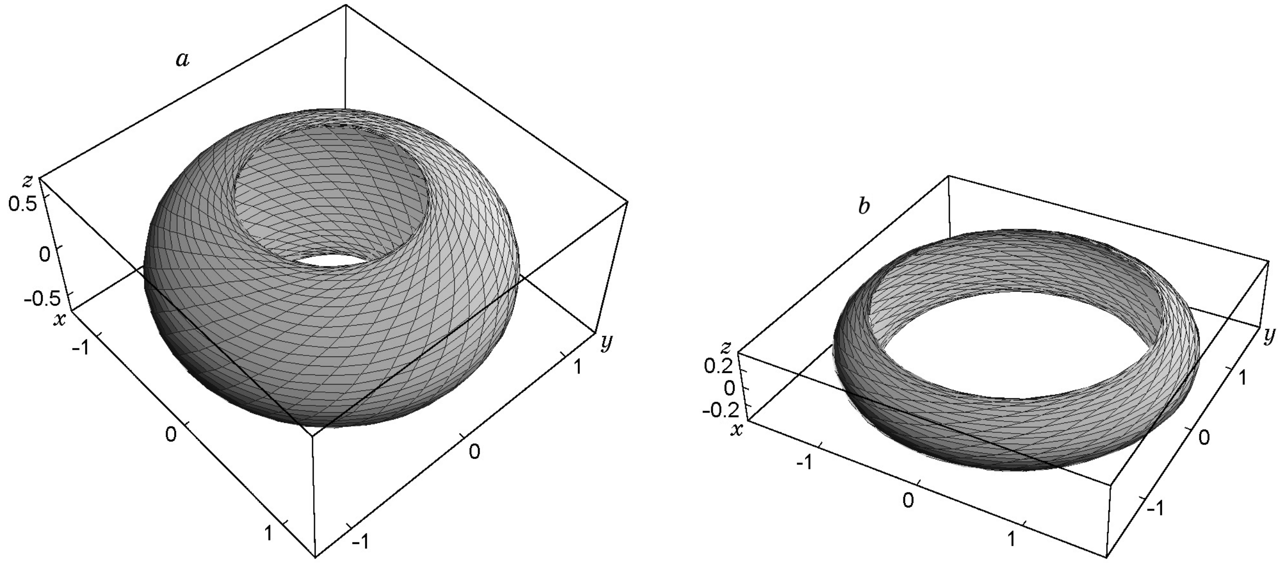

Fig. 4

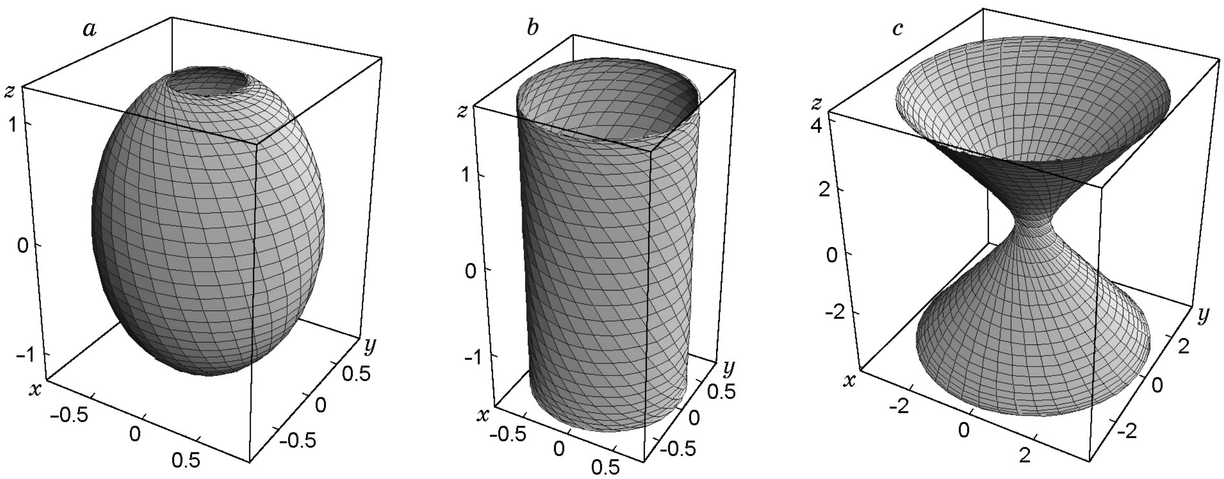

Fig. 5

In $\mathbb{R}^3(\boldsymbol{x})$, relation (20) defines the surface of revolution whose axis passes through the points $O(0,0,0)$ and $P_0(x_0,y_0,z_0)$, and the distance from the generatrix of the surface to the axis is $R(z) = \sigma|z_0|||z/z_0|^{1/2} - z/z_0|$. The trajectory of the fluid particles is obtained when the plane (19) intersects the surface (20). The singularities of this surface [as follows from Eq. (20)] are the degeneration into a point on the axis at $z = z_0$, contraction (neck) at $|z/z_0|^{1/2} = -\sqrt[3]{2}$, asymptotic approach to the plane $z = 0$, and spreading over this plane from two sides as $z \to \pm 0$ (flattening).

We consider the dynamics of a fluid sphere bounded by the sphere $x_0^2 + y_0^2 + z_0^2 = a^2$ at $t_0 = 0$. On the sphere with radius a, we introduce the coordinates θ and φ. Then, the evolution of fluid particles of the sphere is defined by the equations

$$x = a[\tau^{-1}\sin\theta\cos\varphi + (\sigma/3)(\tau^{-1} - \tau^2)\cos\theta\sin\varphi],$$
$$y = a[\tau^{-1}\sin\theta\sin\varphi + (\sigma/3)(\tau^{-1} - \tau^2)\cos\theta\cos\varphi], \qquad z = a\tau^2\cos\theta. \tag{21}$$

Since the coordinates (21) depend linearly on the radius a, the motion of the fluid particles occurs layer-by-layer. All points of the fluid sphere, except for those lying on the axis Oz_0, have nonzero radial and circumferential velocity components. The particles lying on the axis Oz_0 move along this axis to the center of the sphere. The vortex of the initial velocity field (17) has the form (also in cylindrical coordinates) $\boldsymbol{\omega}_0 = (-\sigma, 0, \sigma r^{-1} z_0)$ such that ω^3 becomes infinite on the sphere axis. Such specificity of the initial velocity field leads to violation of the connectivity of the region during motion. It is easy to show at any time that Eqs. (21) describe a torus-shaped body that has an internal cavity.

A consequence of relations (21) is the boundary equation in the form

$$\tau^2(x^2 + y^2) + \tau^{-4} b(\tau) z^2 = a^2, \tag{22}$$

where $b(\tau) = 1 - (\sigma^2/9)(1 - \tau^3)^2$. As follows from Eq. (22), the torus-shaped surface evolves, and at various times, it can be an ellipsoid with a cavity ($b > 0$), a hollow cylinder ($b = 0$), and a one-cavity hollow hyperboloid ($b > 0$).

Figure 4 shows the evolution of the sphere with approach to the singular point $t = 1$ at times $t = 0.2$ (a) and 0.6 (b). Figure 5 shows various stages of the sphere dynamics for $t < 0$: an ellipsoid at $t = -0.1$ (a), a hollow cylinder at $t = 1 - \sqrt[3]{2}$ (b), and a hyperboloid at $t = -0.4$ (c). Singularities of the motion are the degeneration of dimensionality of the region occupied by the fluid and the flattening of the sphere with time. As in the previous examples, the volume of this region does not change with time.

The question of the existence of such solutions is connected with the problem of the applicability limits of the ideal incompressible fluid model.

Conclusions. In the case of a multiple eigenvalue of the matrix $J = \partial \boldsymbol{u}/\partial \boldsymbol{x}$, solutions of the Euler equations (1) with quadratic pressure in the form of (2) form a broad class described by simple formulas and characterized by the nontrivial geometry and physics of motion. Such motions always have a singularity which is manifested in flattening or stretching of the region occupied by the fluid.

Examples of evolution are given for a fluid volume that has a spherical shape at the initial time. The initial velocity field is given for which the connectivity of the region is violated during motion.

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